

Q-CURVATURE OF WEYL STRUCTURES AND POINCARÉ METRICS

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ABSTRACT. We study an asymptotic Dirichlet problem for Weyl structures on asymptotically hyperbolic manifolds. By the bulk-boundary correspondence, or more precisely by the Fefferman–Graham theorem on Poincaré metrics, this leads to a natural extension of the notion of Branson’s Q -curvature to Weyl structures on even-dimensional conformal manifolds.

INTRODUCTION

Let $\overline{X} = X \sqcup \partial X$ be a smooth compact manifold-with-boundary of dimension $n + 1$, and g a smooth conformally compact metric on X , i.e., a Riemannian metric for which $r^2 g$ extends to a smooth metric \overline{g} on \overline{X} , where $r \in C^\infty(\overline{X})$ is any boundary defining function. The metric g is called *asymptotically hyperbolic* (abbreviated as AH) if it moreover satisfies $|dr|_{\overline{g}} = 1$ on ∂X . Such a pair (X, g) is a generalization of the ball model of the hyperbolic space \mathbb{H}^{n+1} . The *conformal infinity* of (X, g) is the boundary $M = \partial X$ equipped with the conformal class \mathcal{C} determined by $\overline{g}|_{TM}$, which is independent of r .

In this article, we introduce the notion of the Q -curvature of Weyl structures on (M, \mathcal{C}) through studying a Dirichlet-type problem for Weyl structures on $(\overline{X}, \overline{\mathcal{C}})$, where $\overline{\mathcal{C}}$ is the conformal class of \overline{g} . Our work is a generalization of Fefferman–Graham’s characterization [8] of Branson’s Q -curvature [5].

By definition, a *Weyl structure* (or a *Weyl connection*) ∇ on (M, \mathcal{C}) is a torsion-free linear connection on M that preserves the class \mathcal{C} . If we pick any representative metric $h \in \mathcal{C}$ as a “reference metric” and let ∇^h be the associated Levi-Civita connection, then a torsion-free linear connection ∇ is a Weyl structure if and only if it satisfies $\nabla = \nabla^h + \beta$ for some (unique) 1-form $\beta \in \Omega^1(M)$, meaning $\nabla h = -2\beta \otimes h$, or equivalently

$$\nabla_\xi \eta = \nabla_\xi^h \eta + \beta(\xi)\eta + \beta(\eta)\xi - h(\xi, \eta)\beta^\sharp,$$

where β^\sharp is the metric dual of β . If $h' = e^{2\Upsilon} h \in \mathcal{C}$ is another representative, where $\Upsilon \in C^\infty(M)$, then the 1-form β' satisfying $\nabla = \nabla^{h'} + \beta'$ is given by $\beta' = \beta - d\Upsilon$. Therefore, a Weyl structure $\nabla = \nabla^h + \beta$ is a Levi-Civita connection if and only if β is exact, and is locally a Levi-Civita connection if and only if β is closed. In the latter case, we also say that ∇ itself is *closed*.

Suppose (X, g) is given, and let $\overline{\nabla}$ be a Weyl structure on $(\overline{X}, \overline{\mathcal{C}})$. As $\overline{\nabla}$ may not be a Levi-Civita connection, its curvature tensor does not necessarily satisfy the usual Riemannian symmetry properties. In particular, the Ricci tensor is not symmetric in general. We call the skew-symmetric part of $\text{Ric}_{\overline{\nabla}}$ the *Faraday tensor* $F_{\overline{\nabla}}$. It is known that, if $\overline{g} \in \overline{\mathcal{C}}$ is any representative and $\overline{\nabla} = \nabla^{\overline{g}} + \overline{b}$, then $F_{\overline{\nabla}}$ equals a constant times $d\overline{b}$ (the constant being dependent on convention). Consequently, the Faraday tensor $F_{\overline{\nabla}}$ determines $\overline{\nabla}$ up to addition of a closed 1-form.

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We consider the following curvature constraint, which is the Euler–Lagrange equation for the Lagrangian density $|F_{\overline{\nabla}}|_g^2$:

$$(0.1) \quad d_g^* F_{\overline{\nabla}} = 0.$$

We have a canonical reference metric for $\overline{\nabla}$ on X , which is the metric g . By putting $\overline{\nabla} = \nabla^g + b$, we can reformulate (0.1) into an equation for a 1-form $b \in \Omega^1(X)$, which is known as the (massless) Proca equation:

$$(0.2) \quad d_g^* db = 0.$$

Since $F_{\overline{\nabla}}$ is invariant under the change $\overline{\nabla} \rightsquigarrow \overline{\nabla} + \gamma$ for any closed 1-form $\gamma \in \Omega^1(\overline{X})$, so is equation (0.2). To break this gauge invariance as much as possible, we introduce the Feynman gauge condition:

$$(0.3) \quad d_g^* b = 0.$$

Then clearly, the solutions of the system of equations (0.2) and (0.3) have only the freedom of adding harmonic 1-forms.

The natural Dirichlet data for Weyl structures $\overline{\nabla}$ on $(\overline{X}, \overline{\mathcal{C}})$ are given by those on (M, \mathcal{C}) ; note that the notion of the induced Weyl structure on M by $\overline{\nabla}$ makes sense because $\overline{\mathcal{C}}$ determines the orthogonal decomposition $(T\overline{X})|_M = TM \oplus T^\perp M$. The Dirichlet problem for our system of equations can be solved as follows.

Theorem 0.1. *Let n be even and $n \geq 4$. Suppose that g is an AH smooth conformally compact metric on X , and let ∇ be a smooth Weyl structure on the conformal infinity (M, \mathcal{C}) , where $M = \partial X$. Then there exists a C^{n-3} Weyl structure $\overline{\nabla}$ on \overline{X} with induced Weyl structure ∇ on M satisfying (0.2) and (0.3). It is unique up to addition of an L^2 -harmonic 1-form on X .*

It is known that any L^2 -harmonic 1-form $\gamma \in \Omega^1(X)$ is smoothly extended to \overline{X} , which is a consequence of the fact that γ admits a “polyhomogeneous expansion” and its logarithmic term coefficients all vanish since $\gamma|_{TM} = 0$ (see Proposition 1.3 and [1, Section 3.1.1]). Therefore, adding L^2 -harmonic 1-forms does not break the C^{n-3} boundary regularity of $\overline{\nabla}$.

We made an assumption on n in the theorem above because this is the case of our main interest. However, the following theorem for $n \geq 3$ odd can be proved almost by the same argument. Again, L^2 -harmonic 1-forms are smooth up to the boundary.

Theorem 0.1’. *Let n be odd and $n \geq 3$, and $(X, g), \nabla$ as in Theorem 0.1. Then there exists a smooth Weyl structure $\overline{\nabla}$ on \overline{X} with induced Weyl structure ∇ on M satisfying (0.2) and (0.3). It is unique up to addition of an L^2 -harmonic 1-form on X .*

We do not have similar results for $n = 1, 2$ because Mazzeo’s work [14], which gives the analytic basis to our argument, does not apply in these dimensions.

Now let n be even and $n \geq 4$. We next focus on the obstruction to the smoothness of $\overline{\nabla}$ to get a quantity that is conformally invariantly assigned to ∇ , as Graham and Zworski [12] did for functions to characterize the GJMS operators [10]. For our purpose, g should be canonically determined to a sufficient order only by the conformal class \mathcal{C} . Hence we take the *Poincaré metric* of Fefferman–Graham [7, 9], which satisfies

$$\text{Ric}_g = -ng + O(r^n) \quad \text{and} \quad \text{tr}_g(\text{Ric}_g + ng) = O(r^{n+2}) \quad \text{at } \partial X.$$

(The first condition means that $|\text{Ric}(g) + ng|_g = O(r^n)$.) If \mathcal{C} is given, then such a g exists, and is unique up to an $O(r^n)$ error with $O(r^{n+2})$ trace and the action of diffeomorphisms of \overline{X} that restricts to the identity on ∂X . Then the aforementioned obstruction is determined only by the

pair (\mathcal{C}, ∇) . Furthermore, it turns out that it is naturally interpreted as a tractor on M . Let us set up the notation: $\mathcal{E}[w]$ is the density bundle of conformal weight w over M , \mathcal{S} is the standard conformal tractor bundle, $\mathcal{S}[w] = \mathcal{S} \otimes \mathcal{E}[w]$, and $\mathcal{S}^*[w] = \mathcal{S}^* \otimes \mathcal{E}[w]$. For the definition of these bundles, we refer to Bailey–Eastwood–Gover [2] or Eastwood’s expository article [6]. By abuse of notation, the spaces of smooth sections of these bundles are denoted by the same symbols. Then we have the following.

Theorem 0.2. *Let g be the Poincaré metric on X , and ∇ a smooth Weyl structure on (M, \mathcal{C}) . Then there exists a density-weighted standard cotractor $\mathbf{Q}_\nabla \in \mathcal{S}^*[n+1]$ on M , which is locally determined by (\mathcal{C}, ∇) , such that any C^{n-3} extension $\bar{\nabla}$ in Theorem 0.1 is smooth if and only if \mathbf{Q}_∇ vanishes.*

Let $h \in \mathcal{C}$ and $\beta \in \Omega^1(M)$ be such that $\nabla = \nabla^h + \beta$. The choice of h determines a direct sum decomposition $\mathcal{S}^* \cong \mathcal{E}[-1] \oplus \Omega^1[1] \oplus \mathcal{E}[1]$, where $\Omega^1[1] = \Omega^1(M) \otimes \mathcal{E}[1]$. Via this decomposition and the trivialization of the density bundles by h , the tractor \mathbf{Q}_∇ is given by

$$(0.4) \quad \mathbf{Q}_\nabla \stackrel{h}{=} (-1)^{n/2-1} 2^{n-2} (n/2 - 1)!^2 \begin{pmatrix} Q_0 1 + G_1 \beta & L_1 \beta & 0 \end{pmatrix}.$$

Here we used the Branson–Gover operators [4] $L_1: \Omega^1(M) \rightarrow \Omega^1(M)$, $G_1: \Omega^1(M) \rightarrow C^\infty(M)$ and $Q_0: C^\infty(M) \rightarrow C^\infty(M)$ (adopting the normalization of Aubry–Guillarmou [1]). In particular,

$$Q_0 1 = \frac{(-1)^{n/2-1}}{2^{n-2} (n/2 - 1)!^2} Q_h,$$

where Q_h is Branson’s Q -curvature of h . Since it is known that L_1 and G_1 annihilate closed forms (see [4]), \mathbf{Q}_∇ is essentially Branson’s Q -curvature when ∇ is a Levi-Civita connection. The authors propose to call \mathbf{Q}_∇ the *Q-curvature tractor* of the Weyl structure ∇ .

For given ∇ , we consider the natural pairing of \mathbf{Q}_∇ and another canonical tractor $\mathbf{W}_\nabla \in \mathcal{S}[-1]$ associated to ∇ . By using any metric $h \in \mathcal{C}$ and $\beta \in \Omega^1(M)$ for which $\nabla = \nabla^h + \beta$, we define

$$\mathbf{W}_\nabla \stackrel{h}{=} \begin{pmatrix} 1 \\ -\beta^\sharp \\ \frac{1}{2}|\beta|^2 \end{pmatrix}.$$

Then the pairing $Q_\nabla = \langle \mathbf{Q}_\nabla, \mathbf{W}_\nabla \rangle \in \mathcal{E}[n]$ can be integrated. Since

$$Q_\nabla / dV_h = Q_h + (-1)^{n/2-1} 2^{n-2} (n/2 - 1)!^2 (G_1 \beta - \langle L_1 \beta, \beta \rangle),$$

we may use the fact that $G_1 \beta$ is the divergence of some 1-form to conclude that, for M compact, Q_∇ integrates to the following global invariant of (M, \mathcal{C}, ∇) :

$$(0.5) \quad \int_M Q_h dV_h + (-1)^{n/2} 2^{n-2} (n/2 - 1)!^2 \int_M \langle L_1 \beta, \beta \rangle dV_h.$$

This can be seen as a functional in the space of Weyl structures on (M, \mathcal{C}) . As the first term, the total Q -curvature, is an invariant of \mathcal{C} , the formula above makes us curious about the spectrum of L_1 . There are explicit formulae for $n = 4$ and 6 [1, Section 8]:

$$L_1 = \frac{1}{2} d^* d \quad (n = 4), \quad L_1 = -\frac{1}{16} d^* \left(\Delta_h - \text{Ric} + \frac{2}{5} \text{Scal} \right) d \quad (n = 6).$$

Here Ric acts as an endomorphism. In four dimensions, this implies that the second term in (0.5) is nonnegative and vanishes if and only if β , or equivalently ∇ , is closed. Hence the integral of Q_∇ minimizes at closed Weyl structures. The same is true in six dimensions under some assumption on the Ricci tensor. In general dimensions, a formula of L_1 can be obtained for an

Einstein metric h by using the idea in third author's article [13]. If $\text{Ric}_h = 2\lambda(n-1)h$ so that the Schouten tensor is $P_h = \lambda h$,

$$(0.6) \quad L_1 = \frac{(-1)^{n/2}}{2^{n-3}(n/2-1)!(n/2-2)!} d^* \left(\prod_{m=1}^{n/2-2} (\Delta_h - 2m(m-n+3)\lambda) \right) d.$$

One may conclude by this that, if \mathcal{C} contains an Einstein metric with positive scalar curvature, then the integral of Q_∇ minimizes exactly at Levi-Civita connections (note that Bochner's Theorem assures the vanishing of $H^1(M)$).

Our theorems are applications of the previous results on the Dirichlet problems for functions and differential forms on AH manifolds. The analytic aspect is due to Mazzeo–Melrose [15] and Mazzeo [14], while the asymptotic expansions were investigated thoroughly by Graham–Zworski [12] and Aubry–Guillarmou [1]. A direct connection to Branson's Q -curvature was found by Fefferman–Graham [8]. In Section 1, we recall their results that are necessary here. We prove our main theorems in Section 2, and the proof of (0.6) is given in Section 3. (For our analysis of Q_∇ , formal asymptotic expansions suffice our needs and the deep results of [15, 14] are not really necessary. However we choose to use them for a clearer exposition.) We shall concentrate on the case where n is even and leave the proof of Theorem 0.1' to the interested reader.

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1. PRELIMINARIES: DIRICHLET PROBLEM FOR FUNCTIONS AND 1-FORMS

We always assume that n is even and $n \geq 4$ in the sequel. Let g be an AH smooth conformally compact metric on X . It is well known [11, Section 5] that a sufficiently small open neighborhood \mathcal{U} of $M \subset \overline{X}$ can be identified with the product $M \times [0, \varepsilon)$ so that

$$(1.1) \quad g = \frac{dx^2 + h_x}{x^2},$$

where x is the coordinate on the second factor of $M \times [0, \varepsilon)$ and h_x is a smooth 1-parameter family of Riemannian metrics on M . The metric $h = h_0$ is a representative of the conformal class \mathcal{C} . In fact, for any prescribed $h \in \mathcal{C}$, there is such an identification; moreover, h determines the identification near ∂X . We call the expression (1.1) the *normalization* of g , and x the *normalizing boundary defining function* of \overline{X} , with respect to h .

We shall summarize fundamental results on the Dirichlet problems for functions and 1-forms. In the original papers, some of them are stated under (weak or genuine) Einstein conditions, but they are actually valid in the following general setting. Asymptotic expansions in the propositions below are given with respect to the identification $\mathcal{U} \cong M \times [0, \varepsilon)$ associated to some fixed h .

Proposition 1.1 (Mazzeo–Melrose [15], Graham–Zworski [12]). *For any function $\varphi \in C^\infty(M)$, there exists a unique harmonic function $\bar{f} \in C^{n-1}(\overline{X})$ with boundary value φ . It has the following*

expansion at the boundary:

$$\bar{f} = \varphi + \sum_{k=1}^{n-1} x^k \varphi_k + x^n \log x \cdot L_0 \varphi + O(x^n), \quad \varphi_k \in C^\infty(M).$$

Here L_0 is a linear differential operator locally determined by g and h , and \bar{f} is smooth if $L_0 \varphi$ vanishes. If g is the Poincaré metric, L_0 is the GJMS operator of critical order up to normalization.

The solvability of the Dirichlet problem and the appearance of the first logarithmic term at the power x^n are consequences of the fact that the characteristic exponents of the Laplacian on functions are 0 and n : Δ_g on functions is expressed as

$$(1.2) \quad \Delta_g = -(x\partial_x)^2 + nx\partial_x + xR,$$

in which R is a polynomial of vector fields that are tangent to ∂X .

A similar technique was used to obtain the following “direct” characterization of Branson’s Q -curvature in terms of the Poincaré metric.

Proposition 1.2 (Fefferman–Graham [8]). *For any representative metric $h \in \mathcal{C}$ and the associated normalizing boundary defining function x , there exists a unique function ρ such that $u = \log \rho - \log x \in C^{n-1}(\bar{X})$, $\log \rho$ is harmonic, and $u|_{\partial X} = 0$. It has the following expansion:*

$$\log \rho = \log x + \sum_{k=1}^{n-1} x^k r_k + x^n \log x \cdot s + O(x^n), \quad r_k, s \in C^\infty(M).$$

The function u is smooth if s vanishes. If g is the Poincaré metric, then

$$s = \frac{(-1)^{n/2-1}}{2^{n-1}(n/2)!(n/2-1)!} Q_h,$$

where Q_h is Branson’s Q -curvature of h .

The corresponding problem for differential forms is studied in [14, 1]. Though differential forms of general degrees are considered in these works, we only use the 1-form case. For a later need, we state the result for general inhomogeneous equations, which also follows from their approach.

Proposition 1.3 (Mazzeo [14], Aubry–Guillarmou [1]). *Let $\bar{a} \in \Omega^1(\bar{X})$ be a smooth 1-form on \bar{X} such that $\bar{a}|_{TM} = 0$. Then for any 1-form $\beta \in \Omega^1(M)$, there exists a solution $\bar{b} \in C^{n-3}(\bar{X}, T^*\bar{X})$ to the equation $\Delta_g \bar{b} = \bar{a}$ satisfying $\bar{b}|_{TM} = \beta$, which is unique modulo L^2 -harmonic 1-forms. It allows the expansion*

$$\bar{b} = \beta + \sum_{k=1}^{n-3} x^k \beta_k + x^{n-2} \log x \cdot \beta^{(1)} + \left(\sum_{k=0}^{n-2} x^k \varphi_k + x^{n-1} \log x \cdot \varphi^{(1)} \right) dx + O^+(x^{n-2}),$$

where $\beta_k, \beta^{(1)} \in \Omega^1(M)$, $\varphi_k, \varphi^{(1)} \in C^\infty(M)$ and the remainder $O^+(x^{n-2})$ is an $O(x^{n-2})$ term that becomes $O(x^{n-1})$ when contracted with ∂_x . The solution \bar{b} is smooth if $\beta^{(1)}$ and $\varphi^{(1)}$ both vanish.

If $\bar{a} = 0$, then there are linear differential operators L_1 and G_1 locally determined by g and h for which $\beta^{(1)} = L_1 \beta$, $\varphi^{(1)} = G_1 \beta$. Moreover, if g is the Poincaré metric, then L_1 and G_1 are the Branson–Gover operators up to normalization.

2. PROOF OF MAIN THEOREMS

Let ∇ be a Weyl structure on (M, \mathcal{C}) . As explained in Introduction, the construction of the extension $\bar{\nabla}$ in Theorem 0.1 boils down to a Dirichlet problem on 1-forms. However, in order to apply Proposition 1.3 for this purpose, g is not appropriate as a reference metric for $\bar{\nabla}$. Indeed, since g diverges at ∂X , so does the 1-form b satisfying $\bar{\nabla} = \nabla^g + b$.

A good choice of reference metric is $\bar{g} = \rho^2 g$, where ρ is the function given in Proposition 1.2 for some $h \in \mathcal{C}$. Since ρ is a (possibly non-smooth) defining function, \bar{g} is a metric on \bar{X} that represents $\bar{\mathcal{C}}$. If we take the 1-form \bar{b} for which $\bar{\nabla} = \nabla^{\bar{g}} + \bar{b}$, then since $\bar{b} = b - d \log \rho$ and $\Delta_g \log \rho = 0$, (0.2) and (0.3) are equivalent to $d_g^* d\bar{b} = 0$ and $d_g^* \bar{b} = 0$. Obviously, for this system to be satisfied, it is necessary that

$$(2.1) \quad \Delta_g \bar{b} = 0.$$

The converse holds actually. In fact, if $\Delta_g \bar{b} = 0$ then $\Delta_g(d_g^* \bar{b}) = 0$ follows. By the conformal change law of the divergence (see Besse [3, 1.159 Theorem]), $d_g^* \bar{b} = \rho^2 d_{\bar{g}}^* \bar{b} + (n-1)\rho \langle d\rho, \bar{b} \rangle_{\bar{g}}$ is continuous up to the boundary and vanishes on ∂X , so the maximum principle implies that $d_g^* \bar{b} = 0$. Hence we also have $d_g^* d\bar{b} = 0$.

Proof of Theorem 0.1. Take an arbitrary pair (h, β) so that $\nabla = \nabla^h + \beta$. We define $\bar{g} = \rho^2 g$, where ρ is the function in Proposition 1.2 associated to h . Then by Proposition 1.3, there is a 1-form $\bar{b} \in C^{n-3}(\bar{X}; T^* \bar{X})$ such that $\Delta_g \bar{b} = 0$ and $\bar{b}|_{TM} = \beta$. We set

$$\bar{\nabla} = \nabla^{\bar{g}} + \bar{b}.$$

Then (0.2) and (0.3) follow because $\Delta_g \bar{b} = 0$ holds. Moreover, for any vector fields $\xi, \eta \in \mathfrak{X}(\bar{X})$ that are tangent to ∂X , the tangential component of $\bar{\nabla}_\xi \eta$ is $\nabla_\xi^h \eta + \beta(\eta)\xi + \beta(\xi)\eta - h(\xi, \eta)\beta^\sharp$, which is $\nabla_\xi \eta$. In this construction, there is an ambiguity in \bar{b} that lies in the L^2 -kernel of Δ_g on 1-forms. Since $\bar{b}|_{TM} = \beta$ is necessary in order that $\bar{\nabla}$ induces ∇ , there is no other ambiguities. \square

It is interesting to see directly that another choice (h', β') would lead to the same Weyl structure $\bar{\nabla}$ (modulo, of course, L^2 -harmonic 1-forms). If $\nabla = \nabla^h + \beta = \nabla^{h'} + \beta'$, then we can write $h' = e^{2\Upsilon} h$ and $\beta' = \beta - d\Upsilon$ by some $\Upsilon \in C^\infty(M)$. Let $\bar{\Upsilon}$ be the harmonic extension of Υ , which uniquely exists by Proposition 1.1. Then the function ρ' in Proposition 1.2 associated to h' is $\rho' = e^{\bar{\Upsilon}} \rho$, and hence $\bar{g}' = \rho'^2 g = e^{2\bar{\Upsilon}} \bar{g}$. On the other hand, a solution to $\Delta_g \bar{b}' = 0$ and $\bar{b}'|_{TM} = \beta'$ is given by $\bar{b}' = \bar{b} - d\bar{\Upsilon}$. Therefore, $\nabla^{\bar{g}'} + \bar{b}'$ and $\nabla^{\bar{g}} + \bar{b}$ are the same.

Next we discuss the smoothness issue.

Lemma 2.1. *Let $h \in \mathcal{C}$ and $\beta \in \Omega^1(M)$ be such that $\nabla = \nabla^h + \beta$. Then, the Weyl structure $\bar{\nabla}$ in Theorem 0.1 is smooth if and only if*

$$(2.2) \quad L_1 \beta = 0 \quad \text{and} \quad ns + G_1 \beta = 0,$$

where $s \in C^\infty(M)$ is given in Proposition 1.2 and L_1, G_1 are as in Proposition 1.3.

Proof. We take the normalization of the metric g with respect to h , and take the 1-form \tilde{b} so that $\bar{\nabla} = \nabla^{x^2 g} + \tilde{b}$. Then, since $x^2 g$ is smooth up to ∂X , $\bar{\nabla}$ is smooth if and only if \tilde{b} is smooth.

Using ρ and \bar{b} constructed in the proof of Theorem 0.1, \tilde{b} is computed as follows:

$$\begin{aligned}
\tilde{b} &= (d \log \rho + \bar{b}) - d \log x \\
&= d \left(\sum_{k=1}^{n-1} x^k r_k + x^n \log x \cdot s \right) \\
&\quad + \beta + \sum_{k=1}^{n-3} x^k \beta_k + x^{n-2} \log x \cdot L_1 \beta + \left(\sum_{k=0}^{n-2} x^k \varphi_k + x^{n-1} \log x \cdot G_1 \beta \right) dx + O^+(x^{n-2}) \\
&= \beta + \sum_{k=1}^{n-3} x^k (\beta_k + dr_k) + x^{n-2} \log x \cdot L_1 \beta \\
&\quad + \left(\sum_{k=0}^{n-2} x^k (\varphi_k + (k+1)r_{k+1}) + x^{n-1} \log x \cdot (ns + G_1 \beta) \right) dx + O^+(x^{n-2}).
\end{aligned}$$

Therefore, (2.2) is equivalent to that the first logarithmic terms of \tilde{b} being zero; thus (2.2) is necessary for the smoothness. Furthermore, since $\Delta_g \tilde{b} = -\Delta_g d \log x = -d \Delta_g \log x$ and $\Delta_g \log x \in xC^\infty(\bar{X})$ by an explicit computation, it follows that $(\Delta_g \tilde{b})|_{TM} = 0$. Hence by Proposition 1.3, (2.2) is also sufficient. \square

Let us specialize to the case where g is the Poincaré metric. Then, since $ns = Q_0 1$, (2.2) is equivalent to $\mathbf{Q}_\nabla = 0$ if \mathbf{Q}_∇ is defined by (0.4). What remains is to check the well-definedness of \mathbf{Q}_∇ . It is by definition equivalent to that the conformal transformation law of $Q_0 1 + G_1 \beta$ is as follows: if $\hat{h} = e^{2\Upsilon} h$, then

$$(2.3) \quad \hat{Q}_0 1 + \hat{G}_1 \hat{\beta} = e^{-n\Upsilon} (Q_0 1 + G_1 \beta - \langle L_1 \beta, d\Upsilon \rangle).$$

To show this, we recall from [1, Corollary 4.14] that the transformation laws of $Q_0 1$ and G_1 are $\hat{Q}_0 1 = e^{-n\Upsilon} (Q_0 1 + nL_0 \Upsilon)$ and $\hat{G}_1 = e^{-n\Upsilon} (G_1 - \iota_{\text{grad } \Upsilon} L_1)$ (the first one is of course the well-known transformation law of the Q -curvature). We also note that L_1 vanishes on closed forms and $L_0 = (1/n)G_1 d$ (see [1, Proposition 4.12]). So we obtain

$$\begin{aligned}
\hat{G}_1 \hat{\beta} &= e^{-n\Upsilon} (G_1 (\beta - d\Upsilon) - \langle L_1 (\beta - d\Upsilon), d\Upsilon \rangle) \\
&= e^{-n\Upsilon} (G_1 \beta - G_1 d\Upsilon - \langle L_1 \beta, d\Upsilon \rangle) = e^{-n\Upsilon} (G_1 \beta - nL_0 \Upsilon - \langle L_1 \beta, d\Upsilon \rangle).
\end{aligned}$$

Hence (2.3) follows, and the proof of Theorem 0.2 is completed.

3. EXPLICIT COMPUTATION ON CONFORMALLY EINSTEIN MANIFOLDS

In this section, we prove the explicit formula (0.6) of the operator L_1 on a conformally Einstein manifold (M, \mathcal{C}) . The proof here follows the symmetric 2-tensor case carried out in [13]. While the argument in [13] was given in terms of the Fefferman–Graham ambient metric, the same idea can also be implemented by the Poincaré metric, which we adopt in this exposition.

Suppose first that \mathcal{C} does not necessarily carry Einstein representatives. Without losing generality, we may assume that M is the boundary of an $(n+1)$ -dimensional smooth compact manifold-with-boundary \bar{X} . Identify an open neighborhood \mathcal{U} of $M \subset \bar{X}$ with $M \times [0, \varepsilon)$. We fix a representative $h \in \mathcal{C}$ once and for all, and let

$$g = \frac{dx^2 + h_x}{x^2}$$

be a Poincaré metric for which $h_0 = h$ and h_x has an expansion in even powers of x (see [8]).

Recall that, in Proposition 1.3, we called a 1-form $\eta \in \Omega^1(X)$ is $O^+(x^m)$ when η is $O(x^m)$ and $\eta(\partial_x) = O(x^{m+1})$. We now introduce some subspaces of such 1-forms. For each even integer $w \geq -n + 2$, let $\mathcal{A}[w] \subset \Omega^1(X)$ be the space of 1-forms that are expressed, near ∂X , as

$$\eta = x^{-w}\beta_x + x^{-w+2}\varphi_x \frac{dx}{x},$$

where β_x and φ_x are smooth families of 1-forms and functions on M in $x \in [0, \varepsilon)$ with expansions in even powers of x . Moreover, we say that $\eta \in \mathcal{A}[w]$ is in $\mathcal{A}_{\text{df}}[w]$ when $d_g^*\eta = O(x^n)$. Note that $\mathcal{A}_{\text{df}}[-n + 2] = \mathcal{A}[-n + 2]$ (use (3.1) below). For all $w \leq -n$, we set $\mathcal{A}_{\text{df}}[w] (= \mathcal{A}[w])$ to be

$$\left\{ \eta = x^{n-2}\beta_x + x^n\varphi_x \frac{dx}{x} \mid \beta_x \text{ and } \varphi_x \text{ are families as mentioned above such that } \beta_0 = 0 \right\}.$$

We need this somewhat irregular definition for technical reasons which can be seen in the proof of Lemma 3.1. If $\eta \in \mathcal{A}[w]$, we call $\beta = \beta_0 = (x^w\eta)|_{TM} \in \Omega^1(M)$ the *restriction* of η , and η an *extension* of β . It is clear that the restriction of any element in $\mathcal{A}[w]$, $w \leq -n$, is zero.

Consider the following three operators between these spaces:

$$\begin{aligned} E: \mathcal{A}_{\text{df}}[w] &\longrightarrow \mathcal{A}_{\text{df}}[w + 2], & \eta &\longmapsto -\frac{1}{4}\eta, \\ F: \mathcal{A}_{\text{df}}[w] &\longrightarrow \mathcal{A}_{\text{df}}[w - 2], & \eta &\longmapsto (\Delta_g + w(w + n - 2))\eta, \\ H: \mathcal{A}_{\text{df}}[w] &\longrightarrow \mathcal{A}_{\text{df}}[w], & \eta &\longmapsto (w + n/2)\eta. \end{aligned}$$

We make the following observations on these operators.

Lemma 3.1. (1) *The operators E , F , and H above are well-defined and form an \mathfrak{sl}_2 -triple.*

(2) *Any $\beta \in \Omega^1(M)$ can be extended to some $\eta \in \mathcal{A}_{\text{df}}[0]$.*

Proof. The most nontrivial point about (1) is that F maps $\mathcal{A}_{\text{df}}[w]$ into $\mathcal{A}_{\text{df}}[w - 2]$. This can be checked using formulae of Aubry–Guillarmou [1, Equations (2.2), (2.3)]. Namely, if we decompose $\eta \in \mathcal{A}[w]$ into the tangential and normal parts as $\eta = \eta^{(t)} + \eta^{(n)}(dx/x)$, then

$$(3.1) \quad d_g^*\eta = \begin{pmatrix} x^2 d_{h_x}^* & -x\partial_x + n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^{(t)} \\ \eta^{(n)} \end{pmatrix} + O(x^{-w+4}),$$

where the term indicated by $O(x^{-w+4})$ is expanded in even powers of x , and

$$(3.2) \quad \Delta_g\eta = \begin{pmatrix} -(x\partial_x)^2 + (n-2)x\partial_x & 0 \\ 2x^2 d_{h_x}^* & -(x\partial_x)^2 + nx\partial_x \end{pmatrix} \begin{pmatrix} \eta^{(t)} \\ \eta^{(n)} \end{pmatrix} + \mathcal{A}[w - 2].$$

Here $\mathcal{A}[w - 2]$ of course denotes some 1-form that belongs to this space. Let $\eta \in \mathcal{A}_{\text{df}}[w]$. Then it is immediate from (3.2) that $F\eta \in \mathcal{A}_{\text{df}}[w - 2]$ for $w \leq -n + 2$. For $w \geq -n + 4$, observe first that the tangential part of $F\eta$ is $O(x^{-w+2})$. Since $d_g^*\eta = O(x^n)$, (3.1) implies that $x^2 d_{h_x}^* \eta^{(t)} + (w - 2 + n)\eta^{(n)} = O(x^{-w+4})$. Then a little computation shows that the normal part of $F\eta$ is $O(x^{-w+4})$. Hence $F\eta \in \mathcal{A}[w - 2]$ also for $w \geq -n + 4$. The fact that $d_g^*F\eta = O(x^n)$ is clear from (1.2) and $d_g^*F\eta = (\Delta_g + w(w + n - 2))d_g^*\eta$.

The assertion (2) follows easily from (3.1). Details are left to the reader. \square

For our purpose, it is also important to note that an extension of β in (2) can be constructed from the harmonic extension \bar{b} given in Proposition 1.3. Using the notation there, we take

$$\eta = \beta + \sum_{k=1}^{n-3} x^k \beta_k + \left(\sum_{k=1}^{n-1} x^k \varphi_{k-1} \right) \frac{dx}{x}.$$

Then one can check that $\beta_k = 0$ and $\varphi_{k-1} = 0$ for k odd, i.e., $\eta \in \mathcal{A}[0]$ actually. Moreover,

$$(3.3) \quad \eta - \bar{b} = -x^{n-2} \log x \cdot L_1 \beta - x^n \log x \cdot (G_1 \beta) \frac{dx}{x} + O^+(x^{n-2})$$

and \bar{b} admits a polyhomogeneous expansion (see [1]). Since \bar{b} satisfies $d_g^* \bar{b} = 0$ and it is known that $L_1 \beta \in \text{im } d_h^*$, we obtain from (3.3) and (3.1) that $\eta \in \mathcal{A}_{\text{df}}[0]$. We will also need the fact that

$$(3.4) \quad \Delta_g \eta = (n-2)x^{n-2} L_1 \beta + nx^n (G_1 \beta) \frac{dx}{x} + O^+(x^n),$$

which follows from (3.3), (3.2), and the fact that $G_1 \beta \in \text{im } d_h^*$.

Remark 3.2. The three operators are also understood by the ambient metric. Recall from [8, Chapter 4] that the ambient metric is given as $\tilde{g} = s^2 g - ds^2$ in the (x, ξ, s) -coordinates, which are related with the standard (ρ, ξ, t) -coordinates¹ on the ambient space $\tilde{\mathcal{G}} \cong \mathbb{R} \times M \times (0, \infty)$ by

$$x = \sqrt{-2\rho}, \quad s = \sqrt{-2\rho}t$$

in the subdomain $\{\rho < 0\}$. The Poincaré manifold (X, g) can be seen as the hypersurface $\{s = 1\}$ of $\tilde{\mathcal{G}}$. Let $\eta \in \mathcal{A}_{\text{df}}[w]$, and for simplicity, assume that $w \geq -n+2$ and $d_g^* \eta = 0$. Assign to it the 1-form $\tilde{\eta} = s^w \eta$ on $\{\rho < 0\} \subset \tilde{\mathcal{G}}$. Then actually $\tilde{\eta}$ can be extended smoothly across $\rho = 0$, and the restriction of η to M corresponds to the pullback of $\tilde{\eta}$ to $\{\rho = 0, t = 1\}$. Now let $T = s\partial_s$. Then E , F , and H correspond to

$$\tilde{\eta} \mapsto -\frac{1}{4}s^2 \tilde{\eta}, \quad \tilde{\eta} \mapsto \tilde{\Delta} \tilde{\eta}, \quad \text{and} \quad \tilde{\eta} \mapsto (\tilde{\nabla}_T + \frac{n}{2} + 1)\tilde{\eta}.$$

For example, noting that $\tilde{\eta}(T) = 0$, $\iota_T(d\tilde{\eta}) = \mathcal{L}_T \tilde{\eta} = w\tilde{\eta}$, and the fact that $\tilde{g} = e^{2v}(g - dv^2)$ if we put $s = e^v$, by the conformal change law of the Hodge Laplacian we conclude that

$$\Delta_{\tilde{g}} \tilde{\eta} = e^{-2v}(\Delta_{g-dv^2} \tilde{\eta} + (n-2)\iota_T(d\tilde{\eta})) = s^{w-2}(\Delta_g + w(w+n-2))\eta.$$

For general $\eta \in \mathcal{A}_{\text{df}}[w]$, we need to introduce more careful assignment of ambient 1-forms. We omit it here. The case of $w \leq -n$ is not important.

We shall detect $L_1 \beta$ in (3.4) using the commutation relations of E , F , and H as in Graham–Jenne–Mason–Sparling [10]. Note first that (3.4) implies $F\eta = E^{n/2-2}\xi$ with some $\xi \in \mathcal{A}[-n+2] = \mathcal{A}_{\text{df}}[-n+2]$ that restricts to $(-4)^{n/2-2}(n-2)L_1 \beta$. Then we can deduce that

$$\begin{aligned} F^{n/2-1}\eta &= F^{n/2-2}E^{n/2-2}\xi = (-1)^{n/2}(n/2-2)!H(H+1)\cdots(H+n/2-3)\xi + E\mathcal{A}_{\text{df}}[-n] \\ &= (n/2-2)!^2\xi + E\mathcal{A}_{\text{df}}[-n]. \end{aligned}$$

Let $\eta' \in \mathcal{A}_{\text{df}}[0]$ be another extension of β . Then since $\eta - \eta' \in E\mathcal{A}_{\text{df}}[-2]$, it follows that $F^{n/2-1}(\eta - \eta') \in E\mathcal{A}_{\text{df}}[-n]$. In particular, we can conclude that

$$(3.5) \quad \begin{aligned} (\text{the restriction of } F^{n/2-1}\eta') &= (-4)^{n/2-2}(n-2)(n/2-2)!^2 L_1 \beta \\ &= (-1)^{n/2} 2^{n-3} (n/2-1)! (n/2-2)! L_1 \beta \end{aligned}$$

for *any* extension $\eta' \in \mathcal{A}_{\text{df}}[0]$ of β .

Now suppose there is an Einstein representative h satisfying $\text{Ric}(h) = 2(n-1)\lambda h$ in the conformal class \mathcal{C} . In this case, one can take $g = x^{-2}(dx^2 + h_x)$, $h_x = (1 - \frac{1}{2}\lambda x^2)^2 h$ as the Poincaré metric. Since L_1 annihilates the closed forms, by the de Rham–Hodge–Kodaira decomposition, we may assume that $d_h^* \beta = 0$ ($\beta \in \text{im } d_h^*$ can even be assumed, but we do not need it here). Because h_x is conformal to h , we also have $d_{h_x}^* \beta = 0$. This implies that the pullback of β by the projection $M \times [0, \varepsilon) \rightarrow M$ is a divergence-free extension of β .

¹It is even more standard to use x for the coordinates on M , but we use ξ instead as x is already reserved.

We compute the Laplacian on 1-forms of the form $\psi(x)\alpha$, where $\alpha \in \Omega^1(M)$ is divergence-free. By (3.2), $\Delta_g(\psi(x)\alpha)$ is again in this form and

$$\Delta_g(\psi(x)\alpha) = \left(-(x\partial_x)^2 + (n-2)\frac{1 - \frac{1}{2}\lambda x^2}{1 + \frac{1}{2}\lambda x^2}x\partial_x \right) \psi(x)\alpha + \frac{x^2}{(1 - \frac{1}{2}\lambda x^2)^2} \psi(x)\Delta_h\alpha.$$

If we put $y = x(1 - \frac{1}{2}\lambda x^2)^{-1}$, then

$$\Delta_g(\psi(x)\alpha) = \left(-(y\partial_y)^2 + (n-2)y\partial_y - 2\lambda y^2(y\partial_y)^2 + 2(n-3)\lambda y^2 \cdot y\partial_y \right) \psi\alpha + y^2\psi\Delta_h\alpha.$$

Hence, if we take $\psi(x) = y^w$, then $F(y^w\alpha) = y^{-w+2}(\Delta_h - 2\lambda w(w-n+3))\alpha$. By applying this repeatedly, we obtain

$$F^{n/2-1}\beta = y^{n-2} \left(\prod_{w=0}^{n/2-2} (\Delta_h - 2\lambda w(w-n+3)) \right) \beta,$$

which combined with (3.5) gives the formula of $L_1\beta$ for divergence-free 1-forms β . Reformulating it for general 1-forms, we get (0.6).

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